A LOWER BOUND FOR THE EXPONENT OF CONVERGENCE OF NORMAL SUBGROUPS OF KLEINIAN GROUPS

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ABSTRACT. In this note we give a short new proof that for each non-elementary Kleinian group Γ, the exponent of convergence of an arbitrary non-trivial normal subgroup is bounded below by half of the exponent of convergence of Γ, and that strict inequality holds if Γ is of divergence type. Our proof uses the existence of a certain uniformly finite-to-one map from a factor of Γ to the normal subgroup.

1. INTRODUCTION AND STATEMENT OF RESULTS

A $(n+1)$-dimensional hyperbolic manifold is given by the Poincaré disc model $\mathbb{D} := \left\{z \in \mathbb{R}^{n+1} : \|z\| < 1\right\}$ of hyperbolic $(n+1)$-space, $n \in \mathbb{N}$, quotiened by the action of a Kleinian group Γ, that is a discrete subgroup of the group of all isometries of $\mathbb{D}$ with respect to the hyperbolic metric d. For a Kleinian group Γ the Poincaré series $P(\Gamma, s) := \sum_{\gamma \in \Gamma} e^{-sd(0,\gamma(0))}$ is canonically associated for each $s \in \mathbb{R}$. The exponent of convergence $\delta(\Gamma)$ of a Kleinian group Γ is given by the abscissa of convergence of the associated Poincaré series, that is $\delta(\Gamma) := \inf\{s : P(\Gamma, s) < \infty\}$, and Γ is of divergence type if the series $P(\Gamma, \delta(\Gamma))$ diverges. The quantity $\delta(\Gamma)$ carries information about the complexity of the action of Γ at the boundary at infinity $\mathbb{S} := \left\{z \in \mathbb{R}^{n+1} : \|z\| = 1\right\}$ of hyperbolic space. For instance, by the Theorem of Bishop and Jones ([BJ97]), it is known that $\delta(\Gamma)$ is equal to the Hausdorff dimension of the radial limit set of Γ which is an important subset of the limit set of Γ. For references on Kleinian groups, limit sets and the associated hyperbolic manifolds we refer to [Bea83, Mas88, Nic89, MT98, Str06].

It is an interesting task to study how the exponents of convergence $\delta(\widehat{\Gamma})$ and $\delta(\Gamma)$ are related for a non-trivial normal subgroup $\widehat{\Gamma}$ of a Kleinian group Γ. Each normal subgroup $\widehat{\Gamma}$ corresponds to a hyperbolic manifold $\widehat{M} = \mathbb{D}/\widehat{\Gamma}$ which is a normal covering of $M = \mathbb{D}/\Gamma$. In [Bro85] it is shown that if Γ is convex cocompact and $\delta(\Gamma) > n/2$ then $\delta(\widehat{\Gamma}) = \delta(\Gamma)$ if and only if the quotient group $\Gamma/\widehat{\Gamma}$ is amenable. A recent result in [Sta11] provides a new proof of Brooks' result, which shows that the restriction $\delta(\Gamma) > n/2$ can be removed. For a recent account on the interplay between the exponent of convergence, the Hausdorff dimension of the limit set and the convex core entropy of Kleinian groups we refer to [FM11].

In this note we give a short new proof of the following Theorem which states that the exponent of convergence $\delta(\widehat{\Gamma})$ of a non-trivial normal subgroup $\widehat{\Gamma}$ of a non-elementary Kleinian group Γ is bounded below by $\delta(\Gamma)/2$, which complements the results on the coincidence of $\delta(\widehat{\Gamma})$ and $\delta(\Gamma)$. Recall that a Kleinian group is called non-elementary if its limit set consists of more than two elements.

Theorem. *Let $\widehat{\Gamma}$ be a non-trivial normal subgroup of a non-elementary Kleinian group Γ. We then have $\delta(\widehat{\Gamma}) \geq \delta(\Gamma)/2$. If Γ is of divergence type then we have $\delta(\widehat{\Gamma}) > \delta(\Gamma)/2$.*

The first assertion of the Theorem was first proven in [FS04, Theorem 2] using [Mat02, Theorem 6]. The second assertion was proven in [Rob05], and it was independently proven using ergodicity of the geodesic

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flow for convex cocompact Kleinian groups in [BTMT12]. In [BTMT12] it is also shown that the lower bound in the Theorem is sharp in the sense that for certain groups of the first kind Γ there exists a sequence of normal subgroups $(\widehat{\Gamma}_n)$ of Γ such that $\delta(\widehat{\Gamma}_n)$ tends to $\delta(\Gamma)/2$.

The methods of our proof presented in this note are essentially different from the ones used before. Our approach is based on the construction of a uniformly finite-to-one map from a factor of the group Γ to the normal subgroup $\widehat{\Gamma}$. Combining this with elementary hyperbolic geometry the first assertion of the Theorem follows. For the second assertion of the Theorem we make additionally use of Proposition 2.3, which is taken from [MY09, Corollary 4.3] and which is based on a uniqueness property of the Patterson-Sullivan measure for Kleinian groups of divergence type. The construction of a finite-to-one map from Γ to the normal subgroup $\widehat{\Gamma}$ has been used in the author's doctoral dissertation [Jae11b, Theorem 6.2.10] in order to obtain the first assertion of the Theorem for normal subgroups of Fuchsian groups of Schottky type. Furthermore, this idea was used to obtain the analogue of the second assertion of the Theorem in the context of fractal models of normal subgroups of Schottky groups in [Jae11a, Theorem 1.2].

We remark that in certain special cases we are able to simplify the proof of the Theorem even further by replacing the uniformly finite-to-one map (see the definition prior to Lemma 2.2) by a one-to-one map from Γ to $\widehat{\Gamma}$ (see Proposition 2.4). These special cases include the case where Γ is a free group of rank greater than one and $\widehat{\Gamma}$ is an arbitrary non-trivial normal subgroup of Γ, as well as all the non-elementary Kleinian groups Γ and non-trivial normal subgroups $\widehat{\Gamma}$ such that $\widehat{\Gamma}$ contains a free subgroup of rank two which is a malnormal subgroup of Γ. We remark that regarding Fuchsian groups ($n = 1$), all torsion-free Fuchsian groups not corresponding to a closed surface are free groups and therefore covered by Proposition 2.4. Furthermore, by a result of [Kap99] it is known that any non-elementary subgroup of a torsion-free word hyperbolic group Γ contains a free group of rank two which is malnormal in Γ. Thus, the second special case of Proposition 2.4 also covers all convex cocompact Kleinian groups Γ. Using the concept of relative hyperbolicity it is possible to apply Proposition 2.4 to all non-elementary geometrically finite groups Γ. In the case of Fuchsian groups, we have that geometrically finite groups coincide with finitely generated groups, whereas in higher dimension we only have that geometrically finite groups are finitely generated. However, in the case $n = 2$, we can use that for each finitely generated group Γ there exists a geometrically finite group Γ' which is isomorphic to Γ. Thus, Proposition 2.4 is applicable to Γ' and all its non-trivial normal subgroups. Since the result of Proposition 2.4 is in the setting of abstract groups, it also applies to all non-trivial normal subgroups of all non-elementary finitely generated Kleinian groups in dimension $n = 2$. It would be interesting to know if Proposition 2.4 can be extended to all non-elementary Kleinian groups (see Problem 2.5).

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2. Proof of the Theorem

We start with the proof of an elementary lemma which allows us to investigate the Poincaré series in terms of a certain relative Poincaré series. For the hyperbolic metric d and $A \subset \mathbb{D}$ we set $d(0,A) := \inf_{x \in A} d(0,x)$. For a subgroup H of a group G we denote by $H\backslash G$ the set of right cosets $\{Hg : g \in G\}$ and for each $g \in G$ we set $[g] := Hg$.

Lemma 2.1. *Let Γ be a non-elementary Kleinian group. For each hyperbolic element $h \in \Gamma$ and for each $s > 0$ there exists a constant $C > 0$ such that*

$$\sum_{\gamma \in \Gamma} \mathrm{e}^{-sd(0,\gamma(0))} \leq C \sum_{[g] \in \langle h \rangle \backslash \Gamma} \mathrm{e}^{-sd(0,[g](0))}.$$

Proof. Without loss of generality we may assume that the origin is an element of the axis $C(h)$ of the hyperbolic element h which joins the fixed points p and q of h. For each $[g] \in \langle h \rangle \backslash \Gamma$ we choose $g_0 \in [g]$ such that the orthogonal projection P_0 of $g_0(0)$ on $C(h)$ satisfies $d(0,P_0) \leq d(0,h(0))$. We then clearly have

$$d(0,[g](0)) \leq d(0,g_0(0)) \leq d(0,h(0)) + d(P_0,g_0(0)).$$

Using this estimate it follows for each $n \in \mathbb{Z}$ that the orthogonal projection P_n of $h^n g_0(0)$ on $C(h)$, which is given by $P_n = h^n P_0$, satisfies

$$d(P_n, h^n g_0(0)) = d(P_0, g_0(0)) \geq d(0,[g](0)) - d(0,h(0)). \tag{2.1}$$

By the First Law of Cosines ([Rat06, Theorem 3.5.3] for right-angled hyperbolic triangles we can estimate for each $n \in \mathbb{Z}$ that

$$d(0,h^n g_0(0)) + 2\log 2 \geq d(0,P_n) + d(P_n, h^n g_0(0)). \tag{2.2}$$

Combining the previous inequality with the fact that $d(0,P_n) \geq (|n|-1)\, d(0,h(0))$ and using the inequality in (2.1) we conclude for each $n \in \mathbb{Z}$ that

$$\begin{aligned} d(0,h^n g_0(0)) + 2\log 2 &\geq (|n|-1)\, d(0,h(0)) + d(0,[g](0)) - d(0,h(0)) \\ &= (|n|-2)\, d(0,h(0)) + d(0,[g](0)). \end{aligned}$$

Consequently, we obtain for each $s > 0$ and for all $n \in \mathbb{Z}$ that

$$\mathrm{e}^{-sd(0,h^n g_0(0))} \leq 2^{-2s} \mathrm{e}^{-(|n|-2)d(0,h(0))} \mathrm{e}^{-sd(0,[g](0))}.$$

Summing the previous inequality over $n \in \mathbb{Z}$ shows that there exists a constant $C > 0$ depending on $s > 0$ and $h \in \Gamma$ such that for each $[g] \in \langle h \rangle \backslash \Gamma$ we have

$$\sum_{\gamma \in [g]} \mathrm{e}^{-sd(0,\gamma(0))} = \sum_{n \in \mathbb{Z}} \mathrm{e}^{-sd(0,h^n g_0(0))} \leq C \mathrm{e}^{-sd(0,[g](0))}. \tag{2.3}$$

Finally, summing over $[g] \in \langle h \rangle \backslash \Gamma$ in (2.3) finishes the proof of the lemma. □

For each normal subgroup $\widehat{\Gamma}$ of Γ and for each $h \in \widehat{\Gamma}$ we consider the map $\iota_h : \Gamma \to \widehat{\Gamma}$ which is given by $g \mapsto g^{-1}hg$, for each $g \in \Gamma$. Since for each $n \in \mathbb{Z}$ and $g \in \Gamma$ we have $\iota_h(h^n g) = g^{-1}h^{-n}hh^n g = \iota_h(g)$, this defines a map $\iota_h : \langle h \rangle \backslash \Gamma \to \widehat{\Gamma}$. The next lemma shows that this map is uniformly finite-to-one.

Lemma 2.2. *Let $\widehat{\Gamma}$ be a non-trivial normal subgroup of a non-elementary Kleinian group Γ. For each hyperbolic element $h \in \widehat{\Gamma}$ there exists a constant $k \in \mathbb{N}$ such that the map $\iota_h : \langle h \rangle \backslash \Gamma \to \widehat{\Gamma}$ is at most k-to-one.*

Proof. Let p and q denote the fixed points of h and let H denote the subgroup of Γ which preserves the fixed points of h, that is $H := \{g \in \Gamma : g(\{p,q\}) = \{p,q\}\}$. Since the limit set of H is equal to $\{p,q\}$, we have that H is an elementary group containing the hyperbolic element h. Therefore, H is a finite extension of the cyclic group $\langle h_0 \rangle$ generated by some hyperbolic element $h_0 \in H$, and there exists $l \in \mathbb{Z}$ such that $h = h_0^l$. We conclude that $\langle h \rangle$ is a subgroup of H with finite index $[H : \langle h \rangle] = k-1$, for some $k \geq 2$.

We now show that $\iota_h : \langle h \rangle \backslash \Gamma \to \widehat{\Gamma}$ is at most k-to-one. Let $[g_1], \ldots, [g_{k+1}] \in \langle h \rangle \backslash \Gamma$ be given such that $\iota_h([g_1]) = \cdots = \iota_h([g_{k+1}])$. Then for each $j \in \{1, \ldots, k\}$ we have that $g_j^{-1}hg_j = g_{k+1}^{-1}hg_{k+1}$, which implies

that $hg_jg_{k+1}^{-1} = g_jg_{k+1}^{-1}h$. Therefore, for each $j \in \{1,\ldots,k\}$ we have that $g_jg_{k+1}^{-1}$ commutes with h, which implies that the fixed points of h are preserved by $g_jg_{k+1}^{-1}$. Consequently, for each $j \in \{1,\ldots,k\}$ we have that $g_jg_{k+1}^{-1} \in H$. Since $[H : \langle h\rangle] = k-1$ it follows by the pigeonhole principle that there exist distinct integers $m,n \in \{1,\ldots,k\}$ such that $\left[g_mg_{k+1}^{-1}\right] = \left[g_ng_{k+1}^{-1}\right]$ in $\langle h\rangle \backslash H$ and hence, $[g_m] = [g_n]$ in $\langle h\rangle \backslash \Gamma$. The proof is complete.

□

For the sake of completeness we cite the following result from [MY09, Corollary 4.3].

Proposition 2.3. *Let $\widehat{\Gamma}$ denote a Kleinian group of divergence type and let Γ be a Kleinian group which contains $\widehat{\Gamma}$ as a normal subgroup. We then have $\delta(\widehat{\Gamma}) = \delta(\Gamma)$.*

Remark. Matsuzaki and Yabuki have informed the author that they have work in progress which extends the result of Proposition 2.3 to isometry groups of a $\mathrm{CAT}(-1)$ space. By using their result and the construction of a finite-to-one map given in this paper, it seems likely that the lower bound in the Theorem can be extended to the setting of isometry groups of a $\mathrm{CAT}(-1)$ space and their normal subgroups.

We are now in the position to prove the Theorem.

Proof of the Theorem. Since Γ is non-elementary and $\widehat{\Gamma}$ is a non-trivial normal subgroup of Γ, it follows that $\widehat{\Gamma}$ possesses the same limit set, hence $\widehat{\Gamma}$ is also non-elementary (see e.g. [MT98, Lemma 2.2]). Fix a hyperbolic element $h \in \widehat{\Gamma}$ and let $\iota_h : \langle h\rangle \backslash \Gamma \to \widehat{\Gamma}$ denote the map defined prior to Lemma 2.2, which is at most k-to-one by Lemma 2.2, for some $k \in \mathbb{N}$. By Lemma 2.1 there exists a constant $C > 0$ such that

$$\sum_{\gamma\in\Gamma} \mathrm{e}^{-sd(0,\gamma(0))} \leq C \sum_{[g]\in\langle h\rangle\backslash\Gamma} \mathrm{e}^{-sd(0,[g](0))}. \tag{2.4}$$

An application of the triangle inequality shows that for each $g \in \Gamma$ we have

$$\begin{aligned} d\left(0,g^{-1}hg(0)\right) &\leq d\left(0,g^{-1}(0)\right) + d\left(g^{-1}(0),g^{-1}h(0)\right) + d\left(g^{-1}h(0),g^{-1}hg(0)\right) \\ &= d\left(0,g^{-1}(0)\right) + d\left(0,h(0)\right) + d\left(0,g(0)\right) = 2d\left(g(0),0\right) + d\left(h(0),0\right). \end{aligned} \tag{2.5}$$

From this we deduce that for each $[g] \in \langle h\rangle \backslash \Gamma$ we have

$$\mathrm{e}^{-d(0,[g](0))} \leq \mathrm{e}^{d(0,h(0))/2}\mathrm{e}^{-d(0,\iota_h([g])(0))/2}. \tag{2.6}$$

Combining first (2.4) and (2.6) and then using that $\iota_h : \langle h\rangle \backslash \Gamma \to \widehat{\Gamma}$ is at most k-to-one we conclude that

$$\sum_{\gamma\in\Gamma} \mathrm{e}^{-sd(0,\gamma(0))} \leq C\mathrm{e}^{d(0,h(0))/2} \sum_{[g]\in\langle h\rangle\backslash\Gamma} \mathrm{e}^{-sd(0,\iota_h([g])(0))/2} \leq kC\mathrm{e}^{d(0,h(0))/2} \sum_{\rho\in\widehat{\Gamma}} \mathrm{e}^{-sd(0,\rho(0))/2}. \tag{2.7}$$

We can now prove the first assertion of the Theorem. For each $\varepsilon > 0$ we have that $\sum_{\gamma\in\Gamma} \mathrm{e}^{-(\delta(\Gamma)-\varepsilon)d(0,\gamma(0))} = \infty$. Hence, it follows by (2.7) that $\sum_{\rho\in\widehat{\Gamma}} \mathrm{e}^{-(\delta(\Gamma)-\varepsilon)d(0,\rho(0))/2} = \infty$. Since this holds for each $\varepsilon > 0$, we conclude that $\delta(\widehat{\Gamma}) \geq \delta(\Gamma)/2$. For the proof of the second assertion let Γ be of divergence type and suppose for a contradiction that $\delta(\widehat{\Gamma}) = \delta(\Gamma)/2$. Then it follows by (2.7) that $\widehat{\Gamma}$ is of divergence type. Since $\widehat{\Gamma}$ is a normal subgroup of Γ, it follows by Proposition 2.3 that $\delta(\widehat{\Gamma}) = \delta(\Gamma)$. But from this we deduce that $\delta(\Gamma) = \delta(\Gamma)/2$ which implies that $\delta(\Gamma) = 0$. This gives the desired contradiction, since it is well-known that for the non-elementary group Γ we have that $\delta(\Gamma) > 0$ (see e.g. [Bea68]). The proof is complete. □

2.1. **Special cases.** In certain special cases the following proposition can further simplify the proof of the Theorem. The next proposition considers abstract groups and might be of independent interest. Recall that

a subgroup H of a group G is called a malnormal subgroup of G if for each $g \in G$ with $g \notin H$ we have that $(gHg^{-1}) \cap H = \{1\}$.

Proposition 2.4. *Let $\widehat{\Gamma}$ be a non-trivial normal subgroup of a group Γ. If Γ is free of rank greater than one, or if $\widehat{\Gamma}$ contains a free subgroup $H = \langle h_1, h_2 \rangle$ of rank two which is a malnormal subgroup of Γ, then there exists a finite set $F \subset \widehat{\Gamma}$ and a map $\tau : \Gamma \to F$ such that the map $\iota : \Gamma \to \widehat{\Gamma}$, given by $\iota(g) := g\tau(g)g^{-1}$, is one-to-one.*

Before we turn to the proof of the proposition let us remark how it is applied in order to simplify the proof of the Theorem. If Proposition 2.4 is applicable, then using the inequality in (2.5) in the proof of the Theorem we can immediately deduce the following inequality

$$\sum_{g\in\Gamma} \mathrm{e}^{-sd(0,g(0))} \leq \max_{h\in F} \mathrm{e}^{d(0,h(0))/2} \sum_{\rho\in\widehat{\Gamma}} \mathrm{e}^{-sd(0,\rho(0))/2},$$

which is analogous to the inequality in (2.7) above. We can then follow the proof of the Theorem after (2.7) and obtain a proof of the Theorem without using Lemma 2.1 and Lemma 2.2.

Proof of Proposition 2.4. First we consider the case that $\Gamma = \langle \gamma_i : i \in I \rangle$ with $\mathrm{card}(I) > 1$ is free. Set $F := \left\{\gamma_1^{\pm 1}, \gamma_2^{\pm 1}\right\}$ and choose some $h_0 \in \widehat{\Gamma} \setminus \{1\}$. Let h_0 be given by the reduced word $h_0 = \gamma_{\omega_1}^{\varepsilon_1} \cdot \dots \cdot \gamma_{\omega_l}^{\varepsilon_l}$, for some $l \in \mathbb{N}$, $\omega \in I^l$ and $\varepsilon \in \{\pm 1\}^l$. We then define $\tau : \Gamma \to F$ as follows. For $g \in \Gamma$ given by the reduced word $\gamma_{\eta_1}^{\rho_1} \cdot \dots \cdot \gamma_{\eta_m}^{\rho_m}$ with $\eta \in I^m$ and $\rho \in \{\pm 1\}^m$, for some $m \in \mathbb{N}$, we choose $\tau(g) \in F \setminus \left\{\gamma_{\eta_m}^{-\rho_m}, \gamma_{\omega_1}^{-\varepsilon_1}, \gamma_{\omega_l}^{\varepsilon_l}\right\}$. We then have that $\iota(g) \in \widehat{\Gamma}$ is given by the reduced word $\gamma_{\eta_1}^{\rho_1} \cdot \dots \cdot \gamma_{\eta_m}^{\rho_m} \tau(g) \gamma_{\omega_1}^{\varepsilon_1} \dots \gamma_{\omega_l}^{\varepsilon_l} \tau(g)^{-1} \gamma_{\eta_m}^{-\rho_m} \cdot \dots \cdot \gamma_{\eta_1}^{-\rho_1}$, which proves that ι is one-to-one.

Secondly, we consider the case that $\widehat{\Gamma}$ contains a free subgroup $H = \langle h_1, h_2 \rangle$ of rank two which is a malnormal subgroup of Γ. Consider the left coset decomposition of Γ with respect to H, that is $\Gamma = \bigcup_{j\in J} g_j H$, for some index set J and a fixed choice of $(g_j) \in \Gamma^J$. Then for each $g \in \Gamma$ there exist $j \in J$, $l \in \mathbb{N}$, words $\omega \in \{1,2\}^l$ and $\varepsilon \in \{\pm 1\}^l$ such that $g = g_j h_{\omega_1}^{\varepsilon_1} \cdot \dots \cdot h_{\omega_l}^{\varepsilon_l}$ and $h_{\omega_1}^{\varepsilon_1} \cdot \dots \cdot h_{\omega_l}^{\varepsilon_l}$ is reduced. We then set $F := \{h_1, h_2\}$ and define $\tau : \Gamma \to F$ as follows. If $\omega_l = 1$ then we set $\tau(g) := h_2$, otherwise we set $\tau(g) := h_1$. We now verify that the map $\iota : \Gamma \to \widehat{\Gamma}$ given by $g \mapsto g\tau(g)g^{-1}$ is one-to-one. Let $g, g' \in \Gamma$ with $\iota(g) = \iota(g')$ be given. There exists indices $i, j \in J$, integers $l, m \in \mathbb{N}$ and words $\omega \in I^l$, $\eta \in I^m$, $\varepsilon \in \{\pm 1\}^l$ and $\rho \in \{\pm 1\}^m$ such that $g = g_i h$ and $g' = g_j h'$ with reduced words $h = h_{\omega_1}^{\varepsilon_1} \cdot \dots \cdot h_{\omega_l}^{\varepsilon_l}$ and $h' = h_{\eta_1}^{\rho_1} \cdot \dots \cdot h_{\eta_m}^{\rho_m}$. First note that $\iota(g) = \iota(g')$ implies that

$$g_j^{-1} g_i h \tau(g) h^{-1} g_i^{-1} g_j = h' \tau(g') \left(h'\right)^{-1}. \tag{2.8}$$

Note that in here $h\tau(g)h^{-1}$ and $h'\tau(g')(h')^{-1}$ are both elements of H which are not equal to 1 by the definition of τ. Since H is a malnormal subgroup of Γ, the equality in (2.8) implies that $g_j^{-1} g_i \in H$ and thus $i = j$. By (2.8) again we then conclude that $h = h'$, which implies that $g = g_i h = g_i h' = g'$. The proof is complete. □

Problem 2.5. Let $\widehat{\Gamma}$ be a non-trivial normal subgroup of a non-elementary Kleinian group Γ. Does there exist a finite set $F \subset \widehat{\Gamma}$ and a map $\tau : \Gamma \to F$ such that the map $\iota : \Gamma \to \widehat{\Gamma}$ given by $g \mapsto g\tau(g)g^{-1}$ is one-to-one?

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